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Supersymmetry with discrete transformations

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Abstract. Starting from the basic supersymmetry algebra of $2n$-component Weyl spinors, we show how this is enlarged to incorporate the discrete symmetries of parity and charge conjugation. The transformation properties of the superfield representations of the enlarged Dirac supersymmetry are found, and the particle content of a simple example examined.

1. Introduction

The original Wess and Zumino supersymmetry (Wess and Zumino 1974a, b, Salam and Strathdee 1974a, Ferrara *et al* 1974) based on a Majorana spinor, or equivalently a two-component Weyl spinor and its Hermitian conjugate, is most elegantly generalized to include internal symmetries by allowing the Weyl spinor to take on a fundamental representation of the internal SU(n) group (Salam and Strathdee 1974b, Dondi and Sohnius 1974, Wess 1974, Dondi 1975a); the Hermitian conjugate spinor is then of course in the conjugate representation of the internal symmetry group.

Now, the $2n$-component Weyl spinor and its conjugate are only equivalent representations of SU(n) for $n = 2$, and in this case, it is possible to construct 8-component spinors that transform as the fundamental representation of SU(2), and satisfy a Majorana-like constraint. For $n > 2$, it is well known that the inequivalence of the fundamental representation and its conjugate implies that the possibility of finding a Majorana constraint must be abandoned (Salam and Strathdee 1974b, Jenkins 1975) and in order to construct $4n$-component spinors (which are now Dirac spinors) that together with their Hermitian conjugates transform into themselves under charge conjugation and parity, we need to double the number of fundamental Weyl spinors.

In § 2, we enlarge the supersymmetry algebra of Weyl spinors to that of a Dirac supersymmetry to obtain the generalization of the algebra suggested by Salam and Strathdee (1974, 1975), and determine the transformation properties of the superfield representations of this algebra.

Starting from the Weyl spinor algebra, it is easy to see how the U(1)-fermion number group recently discussed by Salam and Strathdee (1975) comes into being with the enlarged Dirac algebra.

Although the superfields have in general 2^{8n} independent field components, in § 3 we show how constraint equations can be applied to limit the superfields and discuss as an example the basic superfield for no internal symmetry.

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2. Dirac supersymmetries

The basic supersymmetry algebra is generated by a $2n$-component Weyl spinor Q^j_α, transforming as an SL(2, C) undotted spinor and as a fundamental n-dimensional representation of SU(n), together with its Hermitian conjugate $\bar{Q}_{\dot{\alpha}j}$, a dotted spinor in the conjugate n-dimensional representation of SU(n) (Wess 1974, Dondi 1975a). The defining algebra can be succinctly given by†

$$[M_{\mu\nu}, Q^i_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})^\beta_\alpha Q^i_\beta \qquad \text{sum over repeated indices} \tag{1a}$$

$$[F_p, Q^j_\alpha] = -\tfrac{1}{2}(\lambda_p)^j_k Q^k_\beta \tag{1b}$$

$$\{Q^j_\alpha, Q^k_\beta\} = [P_\mu, Q^j_\alpha] = 0 \tag{1c}$$

$$\{Q^j_\alpha, \bar{Q}_{k\dot{\beta}}\} = 2(\sigma_\mu)_{\alpha\dot{\beta}}\delta^j_k P^\mu \tag{1d}$$

and the Hermitian conjugates of these equations, together with the Poincaré algebra generated by $M_{\mu\nu}$ and P_μ, and the SU(n) algebra of F_p. It is clear that this algebra can easily be extended to incorporate a simple but important U(1) symmetry which we may wish to include with the algebra above (Dondi 1975b), namely

$$[T^5, Q^i_\alpha] = -\tfrac{1}{2}Q^i_\alpha. \tag{1e}$$

Now, it is well known that in order to construct Dirac spinors, which are themselves fundamental representations of SU(n), the above Weyl spinors are not sufficient and we must introduce a new set R^α_j and $\bar{R}^{\dot{\alpha}j}$ with appropriate Lorentz and SU(n) transformation properties. Then we have as a Dirac spinor‡

$$S^j = \begin{pmatrix} Q^j_\alpha \\ \bar{R}^{\dot{\alpha}j} \end{pmatrix}$$

with

$$\bar{S}_j = S^+_j \gamma_0 = (R^\alpha_j, \bar{Q}_{\dot{\alpha}j}). \tag{2b}$$

The Q spinors (Q^j_α, $\bar{Q}_{\dot{\alpha}j}$) and the R spinors (R^α_j, $\bar{R}^{\dot{\alpha}j}$) are related by charge conjugation and parity such that under charge conjugation

$$S \xrightarrow{\mathrm{C}} S^{\mathrm{C}}_j = C\bar{S}^T_j = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \begin{pmatrix} R^\beta_j \\ \bar{Q}_{\dot{\beta}j} \end{pmatrix} \tag{3a}$$

and under parity

$$S \xrightarrow{\mathrm{P}} S^{\mathrm{P}j} = \gamma_0 S = \begin{pmatrix} 0 & \sigma_0 \\ \bar{\sigma}_0 & 0 \end{pmatrix} \begin{pmatrix} Q^j_\alpha \\ \bar{R}^{\dot{\alpha}j} \end{pmatrix}. \tag{3b}$$

These relations between Dirac spinors and Weyl spinors lead us immediately to the supersymmetry algebra for the R spinors, namely

$$\{R_{j\alpha}, R_{k\beta}\} = [P_\mu, R_{j\alpha}] = 0 \tag{4a}$$

† $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \left(\begin{smallmatrix} 0 & -1 \\ +1 & 0 \end{smallmatrix}\right)$, $(\sigma_\mu)_{\alpha\dot{\beta}} = (1, \sigma_i)$, $(\bar{\sigma}_\mu)^{\dot{\beta}\alpha} = (1, -\sigma_i)$, $\frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu)^\alpha_\alpha = \eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $(\sigma_{\mu\nu})^\beta_\alpha = \frac{1}{2}\mathrm{i}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)_\alpha{}^\beta$ and $(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2}\mathrm{i}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}}$ where σ_i are the Pauli matrices.

‡ In the Weyl representation $\gamma_\mu = \left(\begin{smallmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{smallmatrix}\right)$ and $\gamma^5 = -\gamma_0\gamma_1\gamma_2\gamma_3 = \left(\begin{smallmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{smallmatrix}\right)$.

and

$$\{R_{j\alpha}, \bar{R}^{k}_{\dot\beta}\} = 2(\sigma_\mu)_{\alpha\dot\beta} P^\mu \delta_j^{\ k} \tag{4b}$$

or, equivalently,

$$\{\bar{R}^{k\dot\beta}, R_j^\alpha\} = 2(\bar\sigma_\mu)^{\dot\beta\alpha} P^\mu \, \delta_j^{\ k}, \tag{4c}$$

and we can now obtain the full Dirac supersymmetry from

$$\{S^j, S^{kT}\} = \left\{ \begin{pmatrix} Q^j_\alpha \\ \bar{R}^{\dot\alpha j} \end{pmatrix}, (Q^k_\beta, \bar{R}^{\dot\beta k}) \right\} \tag{5a}$$

and

$$\{S^j, \bar{S}_k\} = \left\{ \begin{pmatrix} Q^j_\alpha \\ \bar{R}^{\dot\alpha j} \end{pmatrix}, (R^\beta_k, \bar{Q}_{\dot\beta k}) \right\}. \tag{5b}$$

The anticommutators $\{Q^j_\alpha, \bar{R}^{\dot\beta k}\}$ and $\{Q^j_\alpha, R^\beta_k\}$ are not so far restricted, however, in general we can set

$$\{Q^j_\alpha, \bar{R}^{\dot\beta k}\} = 0, \tag{6a}$$

since the SU(n) structure is not compatible with saturating the right-hand side with the energy–momentum four vector, a scalar in SU(n) space†, and

$$\{Q^j_\alpha, R^\beta_k\} = \delta^\beta_\alpha \, \delta_j^{\ k} (c_1 Z_1 - \mathrm{i} c_2 Z_2) \tag{6b}$$

where Z_1 and Z_2 are real central charges, with the same dimension as P^μ, that commute with all the other generators of the algebra (Haag *et al* 1975).

Thus, we have for the Dirac supersymmetry

$$\{S^j, S^{kT}\} = 0 \tag{7a}$$

and

$$\{S^j, \bar{S}_k\} = [2(\gamma_\mu) P^\mu + (1) c_1 Z_1 + (\gamma^5) c_2 Z_2] \delta^j_{\ k}. \tag{7b}$$

In the case when $c_1 = 0$, $c_2 = 0$, the algebra of the Dirac supersymmetry is the direct product of two Weyl supersymmetries, in much the same way as SU(3)⊗SU(3) is a direct product, and the transformation properties of Dirac superfields can easily be determined by considering the manifold

$$\phi(x, \alpha^k, \bar\alpha_j) = \mathrm{e}^{-\mathrm{i}Px} \exp(\mathrm{i}\bar\alpha_j S^j + \mathrm{i}\bar{S}_k \alpha^k) \tag{8}$$

where α is a totally anticommuting Dirac spinor:

$$\alpha^k = \begin{pmatrix} \omega^k_\beta \\ \bar\theta^{k\dot\beta} \end{pmatrix} \qquad \bar\alpha_k = (\theta^{\ \beta}_{k}, \bar\omega_{\dot\beta k}), \tag{9}$$

such that

$$\bar\alpha_j S^j + \bar{S}_k \alpha^k = (\theta Q + \bar\omega \bar{R} + R\omega + \bar{Q}\bar\theta) \tag{10}$$

here and henceforth, we leave the summation implicit, e.g. $Px = P^\mu x_\mu$, $\theta Q = \theta^{\ \beta}_{k} Q^k_\beta$ etc.

† For the trivial case of no internal symmetry or for SU(2) symmetry we could have P_μ on the right-hand side, which can easily be seen to be consistent with Majorana constraints in these cases, but we do not wish to complicate matters by considering this choice.

Thus

$$\phi(x, \alpha^k, \bar{\alpha}_k) = \phi(x, \theta, \bar{\theta}, \omega, \bar{\omega}) = e^{-iPx} \exp[i(\theta Q + \bar{Q}\bar{\theta}) + i(\bar{\omega}\bar{R} + R\omega)] \quad (11)$$

and together with this manifold we can define a whole series of manifolds related by shift operations, in exactly the same way as if we were handling a single Weyl manifold. As an example, we have

$$\phi_{12}(x, \alpha, \bar{\alpha}) = e^{-iPx}\, e^{i(\theta Q + \bar{\omega}\bar{R})}\, e^{i(\bar{Q}\bar{\theta} + R\omega)} = e^{-iPx}\, e^{i\bar{\alpha}S}\, e^{i\bar{S}\alpha} \quad (11a)$$

and

$$\phi_{21}(x, \alpha, \bar{\alpha}) = e^{-iPx}\, e^{i(\bar{Q}\bar{\theta} + R\omega)}\, e^{i(\theta Q + \bar{\omega}\bar{R})} = e^{-iPx}\, e^{i\bar{S}\alpha}\, e^{i\bar{\alpha}S} \quad (11b)$$

related to $\phi(x, \alpha, \bar{\alpha})$ by

$$\phi(x, \alpha, \bar{\alpha}) = \phi_{12}(x_\mu + i\bar{\alpha}\gamma_\mu\alpha, \alpha, \bar{\alpha}) = \phi_{21}(x_\mu - i\bar{\alpha}\gamma_\mu\alpha, \alpha, \bar{\alpha}). \quad (12)$$

The transformation properties of superfields under supersymmetry transformations can be obtained in the usual way by considering the left action of a group element

$$G = e^{i(\bar{\beta}S + \bar{S}\beta)}$$

which, for example, leads to the transformation laws of the superfields

$$\phi(x, \alpha, \bar{\alpha}) \xrightarrow{G} \phi(x_\mu - i\bar{\beta}\gamma_\mu\alpha + i\bar{\alpha}\gamma_\mu\beta, \alpha + \beta, \bar{\alpha} + \bar{\beta}) \quad (13a)$$

$$\phi_{12}(x, \alpha, \bar{\alpha}) \xrightarrow{G} \phi(x_\mu + 2i\bar{\alpha}\gamma_\mu\beta + i\bar{\beta}\gamma_\mu\beta, \alpha + \beta, \bar{\alpha} + \bar{\beta}) \quad (13b)$$

$$\phi_{21}(x, \alpha, \bar{\alpha}) \xrightarrow{G} \phi(x_\mu - 2i\bar{\beta}\gamma_\mu\alpha - i\bar{\beta}\gamma_\mu\beta, \alpha + \beta, \bar{\alpha} + \bar{\beta}). \quad (13c)$$

Infinitesimally we have

$$\delta\phi = \left(\eta\frac{\partial}{\partial\theta} + \bar{\eta}\frac{\partial}{\partial\bar{\theta}} + \epsilon\frac{\partial}{\partial\omega} + \bar{\epsilon}\frac{\partial}{\partial\bar{\omega}} + i(\theta\sigma_\mu\bar{\eta} - \eta\sigma_\mu\bar{\theta} + \bar{\omega}\bar{\sigma}_\mu\epsilon - \bar{\epsilon}\bar{\sigma}_\mu\omega)\,\partial^\mu\right)\phi \quad (14a)$$

$$\delta\phi_{12} = \left(\eta\frac{\partial}{\partial\theta} + \bar{\eta}\frac{\partial}{\partial\bar{\theta}} + \epsilon\frac{\partial}{\partial\omega} + \bar{\epsilon}\frac{\partial}{\partial\bar{\omega}} + 2i(\theta\sigma_\mu\bar{\eta} + \bar{\omega}\bar{\sigma}_\mu\epsilon)\,\partial^\mu\right)\phi_{12} \quad (14b)$$

$$\delta\phi_{21} = \left(\eta\frac{\partial}{\partial\theta} + \bar{\eta}\frac{\partial}{\partial\bar{\theta}} + \epsilon\frac{\partial}{\partial\omega} + \bar{\epsilon}\frac{\partial}{\partial\bar{\omega}} - 2i(\eta\sigma_\mu\bar{\theta} + \bar{\epsilon}\bar{\sigma}_\mu\omega)\,\partial^\mu\right)\phi_{21}. \quad (14c)$$

where we have replaced the Dirac spinors by their Weyl equivalents,

$$\beta^k = \begin{pmatrix} \epsilon_\beta^k \\ \bar{\eta}^{k\dot{\beta}} \end{pmatrix}$$

and we see explicitly that the superfields transform as we would expect for direct product representations of two Weyl supersymmetries.

Since ω and θ are independent spinors, the set of Weyl covariant derivatives is doubled and as well as the usual

$$\mathrm{D} = \frac{\partial}{\partial\theta} + i(\sigma_\mu\bar{\theta})\,\partial^\mu, \qquad \bar{\mathrm{D}} = -\frac{\partial}{\partial\bar{\theta}} - i(\theta\sigma_\mu)\,\partial^\mu \quad (15a)$$

acting on ϕ, we have

$$D' = -\frac{\partial}{\partial\omega} - i(\bar{\omega}\bar{\sigma}_\mu)\,\partial^\mu, \qquad \bar{D}' = \frac{\partial}{\partial\bar{\omega}} + i(\bar{\sigma}_\mu\omega)\,\partial^\mu \tag{15b}$$

where

$$\{\bar{D}'^{\dot{\beta}i}, D'^{\alpha}_{j}\} = -2i(\bar{\sigma}_\mu)^{\dot{\beta}\alpha}\,\delta^i_j\,\partial^\mu. \tag{16}$$

Similarly, on ϕ_{12} we have covariant derivatives

$$-\frac{\partial}{\partial\bar{\theta}}, \qquad \frac{\partial}{\partial\theta} + 2i(\sigma_\mu\bar{\theta})\,\partial^\mu, \qquad -\frac{\partial}{\partial\omega}, \qquad \frac{\partial}{\partial\bar{\omega}} + 2i(\bar{\sigma}_\mu\omega)\,\partial^\mu \tag{17a}$$

and on ϕ_{21}

$$\frac{\partial}{\partial\theta}, \qquad -\frac{\partial}{\partial\bar{\theta}} - 2i(\theta\sigma_\mu)\,\partial^\mu, \qquad \frac{\partial}{\partial\bar{\omega}}, \qquad -\frac{\partial}{\partial\omega} - 2i(\bar{\omega}\bar{\sigma}_\mu)\,\partial^\mu. \tag{17b}$$

In each case of course, the Weyl covariant derivatives can be combined in pairs to form Dirac covariant derivatives that satisfy anticommutation relations like equation (16) with γ matrices replacing the Weyl two-dimensional matrices.

We can now go one step further, and determine the transformation rules for the situation where c_1 and c_2 are non-zero. In order to do this, we must assume some form of transformation for the superfields under the Z charges. The simplest choice is to assume

$$[Z_1, \phi] = z_1\phi \qquad [Z_2, \phi] = z_2\phi \tag{18}$$

in which case we find the supersymmetry generators transform the superfields by

$$\phi \xrightarrow{G} \exp[-\tfrac{1}{2}(\bar{\beta}\alpha - \bar{\alpha}\beta)c_1 z_1]\exp[-\tfrac{1}{2}(\bar{\beta}\gamma^5\alpha - \bar{\alpha}\gamma^5\beta)c_2 z_2]\phi' \tag{19a}$$

$$\phi_{12} \xrightarrow{G} \exp(\bar{\alpha}\beta c_1 z_1 + \bar{\alpha}\gamma^5\beta c_2 z_2)\phi'_{12} \tag{19b}$$

$$\phi_{21} \xrightarrow{G} \exp(-\bar{\beta}\alpha c_1 z_1 - \bar{\beta}\gamma^5\alpha c_2 z_2)\phi'_{21} \tag{19c}$$

where the ϕ' are the transformed fields given in equations (13). We should also note the possibility of non-linear transformations, such as

$$[Z, \phi] = K \tag{20}$$

which is also consistent with the algebra and leads to supersymmetry transformations that depend on K, this time the dependence is non-linear. Obviously, the presence of these Z-dependent terms in the supersymmetry transformations also manifests itself in the covariant derivatives, which makes constraints that we may impose on the superfields using the covariant derivatives rather restrictive†.

Finally, we should analyse the content of the U(1) symmetry given in equation (1*e*). As has been shown, in the case of Majorana-type spinors, when a $4n$ spinor is constructed out of Q^j_α and its Hermitian conjugate, this transformation is the γ^5

† Some examples of supersymmetry including Z_1 and Z_2 and an internal SU(2) group have been studied by A Klüver 1975 *Universität Karlsruhe Diplomarbeit* (unpublished).

transformation of the original Wess and Zumino supersymmetry that included the full conformal group and a γ^5 transformation (Wess and Zumino 1974a, Dondi and Sohnius 1974, Dondi 1975b). However, when we define a Dirac spinor by introducing a new set of Weyl spinors, we are left with a certain amount of freedom, namely, under parity and charge conjugation we can require either

$$T^5 \overset{\mathrm{P}}{\to} -T^5 \quad \text{and} \quad T^5 \overset{\mathrm{C}}{\to} T^5 \tag{21a}$$

or

$$T^5 \overset{\mathrm{P}}{\to} T^5 \quad \text{and} \quad T^5 \overset{\mathrm{C}}{\to} -T^5. \tag{21b}$$

For the former choice we have

$$[T^5, \bar{R}^{j\dot{\beta}}] = \tfrac{1}{2}\bar{R}^{j\dot{\beta}} \tag{22}$$

which leads to

$$[T^5, S^j] = -\tfrac{1}{2}(\mathrm{i}\gamma^5)S^j \tag{23}$$

the usual γ^5 transformation, whilst the second choice gives

$$[T^5, \bar{R}^{j\dot{\alpha}}] = -\tfrac{1}{2}\bar{R}^{j\dot{\alpha}} \tag{24}$$

or, equivalently,

$$[T^5, S^j] = -\tfrac{1}{2}S^j \tag{25}$$

a proper U(1) transformation of the complete Dirac spinor. This can play the role of a fermion number U(1) as has recently been suggested by Salam and Strathdee (1975), and we drop the index, five, to avoid confusion between γ^5 transformations and the new U(1) transformations.

Of course, in the case of the γ^5 transformation, the c_1 and c_2 of equation (6*b*) must be zero to satisfy the Jacobi identities.

The superfields transform under the T^5 subgroup as

$$\mathrm{e}^{\mathrm{i}t_5T^5}\phi\, \mathrm{e}^{-\mathrm{i}t_5T^5} = \mathrm{e}^{\mathrm{i}rt_5}\phi(x, \mathrm{e}^{\frac{1}{2}t_5\gamma^5}\alpha, \bar{\alpha}\, \mathrm{e}^{\frac{1}{2}t_5\gamma^5}) \tag{26}$$

for any superfield, although constraint equations which we may impose on the various superfields can limit the possible value of r.

The U(1) group under which S^j transforms as in equation (25) implies a superfield transformation of the form

$$\mathrm{e}^{\mathrm{i}tT}\phi\, \mathrm{e}^{-\mathrm{i}tT} = \mathrm{e}^{\mathrm{i}ft}\phi(x, \mathrm{e}^{\frac{1}{2}\mathrm{i}t}\alpha, \mathrm{e}^{-\frac{1}{2}\mathrm{i}t}\bar{\alpha}) \tag{27}$$

again for any superfield, with the proviso that f may be restricted by constraints on ϕ.

3. An example

The general superfield $\phi(x, \alpha, \bar{\alpha})$, when expanded as a power series in its anticommuting parameters has 2^{8n} independent field components, with high-spin fields being included for higher values of n. Furthermore, the structure of the superfields based on Weyl supersymmetries is well known, and several Lagrangian models have been constructed (see, for example, Capper and Leibbrandt 1975). The simplest extension to include Dirac supersymmetries that we can treat without encountering difficulties due to high-spin fields, is that of a superfield when $n = 1$, i.e. the superfield is just a function of a four-component Dirac spinor and its Hermitian conjugate. We can impose on this superfield the invariant conditions

$$\bar{\mathrm{D}}\phi = 0 \tag{28}$$

and

$$\mathrm{D}'\phi = 0. \tag{29}$$

In the ϕ_{12} representation, these are just

$$\frac{\partial}{\partial \theta}\phi = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \omega}\phi = 0 \tag{30}$$

or, equivalently,

$$\frac{\partial}{\partial \alpha}\phi = 0. \tag{31}$$

From equations (14*b*) and (19*b*), we see that this superfield transforms simply as

$$\begin{aligned}\delta\phi_{12}(x, \bar{\alpha}) &= \delta\phi_{12}(x, \theta, \bar{\omega}) \\ &= \left(\eta\frac{\partial}{\partial \theta} + \bar{\epsilon}\frac{\partial}{\partial \bar{\omega}} + 2\mathrm{i}(\theta\sigma_\mu\bar{\eta} + \bar{\omega}\bar{\sigma}_\mu\epsilon)\,\partial^\mu + \theta\epsilon(c_1 z_1 - \mathrm{i}c_2 z_2) \right. \\ &\quad \left. + \bar{\omega}\bar{\eta}(c_1 z_1 + \mathrm{i}c_2 z_2)\right)\phi_{12}.\end{aligned} \tag{32}$$

Obviously, it does not have a last term, in the power series expansion in θ and $\bar{\omega}$, that transforms as a total divergence unless $c_1 z_1 = c_2 z_2 = 0$ and thus without this condition, we cannot expect to proceed in the usual fashion and obtain an invariant action for this superfield. Of course, it may be possible when there is more than just one superfield to arrange non-zero values of $c_1 z_1$, $c_2 z_2$ or K of different superfields such that their effect cancels, however, at the moment we want to consider the single superfield and set $c_1 z_1 = c_2 z_2 = 0$.

Now, we can consider

$$\phi_{12}(x, \theta, \bar{\omega}) = A_1(x, \theta) + \bar{\Phi}_1^{\dot{\alpha}}(x, \theta)\bar{\omega}_{\dot{\alpha}} - F_1(x, \theta)\bar{w} \tag{33a}$$

with

$$A_1(x, \theta) = A + \theta^\alpha\psi_\alpha + uA' \tag{33b}$$

$$\bar{\Phi}_1^{\dot{\alpha}}(x, \theta) = \bar{\lambda}^{\dot{\alpha}} + \theta^\alpha V^{\dot{\alpha}}{}_\alpha + u\bar{\xi}^{\dot{\alpha}} \tag{33c}$$

$$F_1(x, \theta) = F + \theta^\alpha\chi_\alpha + uF' \tag{33d}$$

where u and $\bar{w}$, the invariants constructed from two θ's and two $\bar{\omega}$'s respectively, are as defined in Dondi (1975a).

We could combine terms together such that ψ_α and $\bar{\lambda}^{\dot{\alpha}}$, and χ_α and $\bar{\xi}^{\dot{\alpha}}$ form two Dirac spinors, but we believe that, computationally, the Weyl notation is easier to handle, and we shall leave the superfield in the form of equation (33). In particular, multiplication of Dirac superfields can be carried out directly or in several steps depending on the form required, thus

$$\begin{aligned}\phi_{12}^n = A_1^n(x, \theta) + nA_1^{n-1}(x, \theta)\bar{\Phi}_1^{\dot{\alpha}}(x, \theta)\bar{\omega}_{\dot{\alpha}} - [nA_1^{n-1}(x, \theta)F_1(x, \theta) \\ - \tfrac{1}{2}n(n-1)A_1^{n-2}(x, \theta)\bar{\Phi}_1^{\dot{\alpha}}(x, \theta)\bar{\Phi}_{1\dot{\alpha}}(x, \theta)]\bar{w}.\end{aligned} \tag{34}$$

The first thing that we should notice is that this superfield is further reducible since we can apply a constraint similar to that used in reducing SU(2) superfields, namely

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\omega}^{(2)}}\mathrm{e}^{2\mathrm{i}(\theta\not\partial\bar{\theta}-\omega\not\partial\bar{\omega})}\phi_{12}(x,\theta,\bar{\omega})=c\,\Box\,\phi_{21}^{+}(x,\omega,\bar{\theta}) \tag{35}$$

where $\mathrm{d}/\mathrm{d}\theta^{(2)}$ and $\mathrm{d}/\mathrm{d}\bar{\omega}^{(2)}$ are the invariant derivative operators as defined in Dondi (1975a) in the ϕ_{21} representation. Equating coefficients of ω_α we find that this constraint is satisfied if

$$cF_1^{+}(x,\bar{\theta})=-4\frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\mathrm{e}^{2\mathrm{i}\,\theta\not\partial\bar{\theta}}A_1(x,\theta) \tag{36a}$$

$$c(\bar{\not\partial})^{\dot{\beta}\beta}\Phi_\beta(x,\bar{\theta})=2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\mathrm{e}^{2\mathrm{i}\theta\not\partial\bar{\theta}}\bar{\Phi}^{\dot{\beta}}(x,\theta) \tag{36b}$$

$$cc^{*}=16. \tag{36c}$$

The constraint equation (35) is reminiscent of that used in constraining an SU(2) superfield (Dondi 1975a, Firth and Jenkins 1974) and this comes about because both the Dirac superfield and the SU(2) superfield depend on two Weyl spinors, whilst (36*b*) is similar to that used previously to constrain a single Weyl spinor superfield (Ferrara *et al* 1974).

Choosing $c=-4$ for convenience, we have† from equation (36*a*)

$$F=A'^{+};\qquad \chi_\alpha=-2\mathrm{i}\,(\not\partial)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}};\qquad F'=-4\,\Box\,A^{+} \tag{37a}$$

and from (36*b*)

$$\bar{\xi}^{\dot{\beta}}=-2\mathrm{i}(\bar{\not\partial})^{\dot{\beta}\beta}\lambda_\beta;\qquad \partial^\nu V_\nu^2=0;\qquad \partial_\mu V_\nu^1-\partial_\nu V_\mu^1=0 \tag{37b}$$

where

$$V_{\alpha\dot{\alpha}}=(\sigma_\mu)_{\alpha\dot{\alpha}}V^\mu=(\sigma_\mu)_{\alpha\dot{\alpha}}(V^{1\mu}+\mathrm{i}V^{2\mu}).$$

Using the constraints in equation (34) for $n=2$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\omega}^{(2)}}\phi_{12}^{2}=2(FF^{+}-4A\,\Box\,A^{+}-2\mathrm{i}\psi\,\not\partial\bar{\psi}-2\mathrm{i}\bar{\lambda}\,\bar{\not\partial}\lambda+\partial_\mu S\,\partial^\mu S-V_\mu^2V^{\mu 2})$$
$$+\text{total divergences} \tag{38}$$

where we have used the constraints on V_μ to replace V_μ^1 by $\partial_\mu S$ and S has canonical dimension 1. This is a candidate for a free massless superfield Lagrangian. We may generate a second Lagrangian different from the first, by using the constraints in an

† These solutions also follow from the more symmetrical constraint

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\omega}^{(2)}}\mathrm{e}^{-2\mathrm{i}\omega\not\partial\bar{\omega}}\phi_{12}=\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(2)}}\mathrm{e}^{-2\mathrm{i}\theta\not\partial\bar{\theta}}\phi_{21}^{+}.$$

inverse manner to give

$$\frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\omega}^{(2)}}\phi_{12}\Box\phi_{12}$$

$$= -\tfrac{1}{2}F'^{+}F'+2A'\Box A'^{+}+\mathrm{i}\bar{\chi}\not{\partial}\chi+\mathrm{i}\xi\not{\partial}\bar{\xi}-2\mathrm{D}^2+(\partial_\mu V^2_\nu-\partial_\nu V^2_\mu)(\partial^\mu V^{2\nu}-\partial^\nu V^{2\mu}) \quad (39)$$

where we have replaced $\partial_\mu V^{\mu 1}$ by D which is here an auxiliary field. Again, this is inherently massless.

The reduction to the simpler superfields can be allowed for in the superfield of equation (33) by replacing

$$F' \rightarrow F' + 4\Box A^+$$

$$\chi_\alpha \rightarrow \chi_\alpha + 2\mathrm{i}(\not{\partial}\bar{\psi})_\alpha$$

$$\bar{\xi}^{\dot\alpha} \rightarrow \bar{\xi}^{\dot\alpha} + 2\mathrm{i}(\bar{\not{\partial}}\lambda)^{\dot\alpha}$$

exactly as in the SU(2) superfield case. It should not now be surprising that a massive Lagrangian can be constructed and that the invariant action takes the form

$$\mathscr{A} = \int \mathrm{d}^4x \frac{\mathrm{d}}{\mathrm{d}\theta^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\bar{\omega}^{(2)}}\frac{\mathrm{d}}{\mathrm{d}\omega^{(2)}}\Big\{\phi^+_{21}(x,\bar{\theta},\omega)\,\mathrm{e}^{2\mathrm{i}(\theta\not{\partial}\bar{\theta}-\omega\not{\partial}\bar{\omega})}\phi_{12}(x,\theta,\bar{\omega})$$

$$+2[\phi_{12}(\Box+2m^2)\phi_{12}\bar{u}w+\textsc{hc}]\Big\}$$

where HC stands for Hermitian conjugate, or, using the correspondence between integration and differentiation for the anticommuting parameters, we can write this as

$$\mathscr{A} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \mathrm{d}^2\omega\, \mathrm{d}^2\bar{\omega}\{\phi^+_{21}\,\mathrm{e}^{2\mathrm{i}(\theta\not{\partial}\bar{\theta}-\omega\not{\partial}\bar{\omega})}\phi_{12}-2[\phi_{12}(\Box+2m^2)\phi_{12}\,\delta(\bar{\theta})\,\delta(\omega)+\textsc{hc}]\}$$

where $\delta(\omega)=w$ etc, and we have replaced differentiation by integration in the standard fashion for the anticommuting parameters. In terms of component fields

$$\mathscr{A} = \int \mathrm{d}^4x[(F'F'^{+}-4A^{+}_{+}\Box A_{+}-2\mathrm{i}\bar{\chi}\,\not{\partial}\chi-2\mathrm{i}\xi\,\not{\partial}\bar{\xi}-8\,V_{\mu\nu}V^{\mu\nu}-16\,V^1_\mu\,\partial^\mu\,\partial_\rho V^{\rho 1})$$

$$-4m^2(A_+A^+_+-A_-A^+_-+16A\,\Box A^++2AF'+2A^+F'^{+}+4V^1_\mu V^{1\mu}$$

$$-4V^2_\mu V^{2\mu}+2\psi^\beta\chi_\beta+2\xi^\alpha\lambda_\alpha+2\bar{\lambda}_{\dot\beta}\bar{\xi}^{\dot\beta}+2\bar{\chi}_{\dot\alpha}\bar{\psi}^{\dot\alpha}+8\mathrm{i}\psi\not{\partial}\bar{\psi}+8\mathrm{i}\bar{\lambda}\bar{\not{\partial}}\lambda)] \quad (40)$$

where $A_\pm = A'^{+} \pm F$ and $V_{\mu\nu}=\partial_\mu V^2_\nu-\partial_\nu V^2_\mu$.

The F' and A_- are auxiliary fields, while the particle content is: two complex spin zero fields, two Dirac spinors, a real scalar and a real vector. If we further require the action to be invariant under a U(1) transformation of the type defined by equation (27), we are forced to choose $f=-1$, and find that the individual fields transform under this U(1) group with f numbers:

$$f_A=-1, \qquad f_\psi=f_{\bar{\lambda}}=-\tfrac{1}{2}, \qquad f_{A'}=f_V=f_F=0$$

$$f_{\bar{\xi}}=f_\chi=\tfrac{1}{2}, \qquad f_{F'}=1$$

exactly as suggested in Salam and Strathdee (1975).

A glance at the expansion of $\phi^n(x, \theta, \bar{\omega})$ indicates that this superfield suffers in the same way as the SU(2) superfield, in that the auxiliary fields introduce a non-polynomial interaction in the component fields for simple polynomial interactions in ϕ^n. On top of this, the added U(1) invariance will also be explicitly broken by adding interactions of the ϕ^n type.

Thus, although the basic Dirac superfield has given us a complete and easily obtainable description of the transformation properties of the component fields, it seems that possibly renormalizable interactions must be introduced in the context of gauge invariance (Salam and Strathdee 1975) and not by direct consideration of the massive Dirac superfield.

References

Capper D M and Leibbrandt G 1975 *Nucl. Phys.* B **85** 503
Dondi P H 1975a *J. Phys. A: Math. Gen.* **8** 1298
—— 1975b *J. Phys. A: Math. Gen.* **8** 1310
Dondi P H and Sohnius M 1974 *Nucl. Phys.* B **81** 317
Ferrara S, Wess J and Zumino B 1974 *Phys. Lett.* **51B** 239
Firth R J and Jenkins J D 1974 *Nucl. Phys.* B **85** 525
Haag R, Łopuszański J T and Sohnius M 1975 *Nucl. Phys.* B **88** 257
Jenkins J D 1975 *J. Phys. A: Math. Gen.* **8** L73
Salam A and Strathdee J 1974a *Nucl. Phys.* B **76** 477
—— 1974b *Nucl. Phys.* B **80** 499
—— 1975 *Nucl. Phys.* B **97** 293
Wess J 1974 *Lecture Notes in Physics* 37 (Berlin: Springer-Verlag)
Wess J and Zumino B 1974a *Nucl. Phys.* B **70** 39
—— 1974b *Phys. Lett.* **49B** 52